

# Polymorphic sessions and sequential composition of types

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PLACES '23

[PLACES '22] Higher - order Context - free Session types in  
System F

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RECAP ON  
CFSTs

- CFSTs are not restricted to tail recursion
- ; sequential composition operator
- SKIP corresponding neutral element
- CFSTs are not characterized by regular languages
- Type equivalence : decidable (via translation to simple grammars)

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$\text{InputTree} \doteq \bigoplus \{ \text{Node} : \text{InputTree} ; ! (? \text{ int}) ; \text{InputTree} ,$   
 $\text{Leaf} : \text{skip} \}$

recursion  
incorporated  
using equations

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$$\forall^S o; !\text{int}$$
$$\forall^T o \rightarrow !\text{int}$$

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- Added polymorphism , using De Bruijn indices to refer to polymorphic variables

$$\forall^s_0; !\text{int}$$

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$$\forall^\tau_0 \rightarrow !\text{int}$$

functional  
 $\kappa := \tau \mid s$   
Kinds      session

[PLACES '22] Higher - order Context - free Session types in  
System F

- But .. we only allowed functional polymorphism !

[PLACES '22] Higher - order Context - free Session types in  
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- We provided: syntactic (rule based) and semantic (bisimulation based) definitions of equivalence;
  - + a type equivalence algorithm by reduction to the bisimilarity of simple grammars .

[PLACES'22] → [PLACES'23]

GOAL · Full polymorphism

$\forall \dots T \rightarrow \text{of Kind } s$

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NEW DESIGN  
CHOICES

## WHAT WE LEARNED

- Using De Bruijn indices restricts expressivity

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$$X = O \rightarrow A^s X$$

equation expansion

$$\hookrightarrow O \rightarrow A^s O \rightarrow A^s X$$

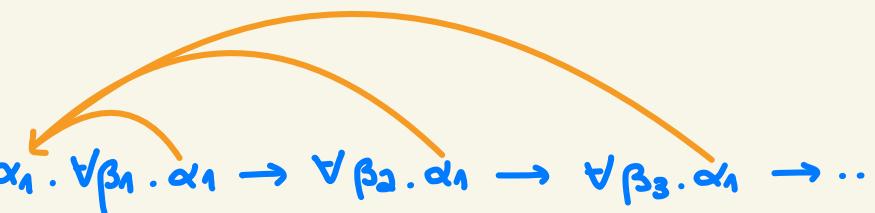
## WHAT WE LEARNED

- Using De Bruijn indices restricts expressivity

$$T \triangleq \forall \alpha_1. \forall \beta_1. \alpha_1 \rightarrow \forall \beta_2. \alpha_1 \rightarrow \forall \beta_3. \alpha_1 \rightarrow \dots$$

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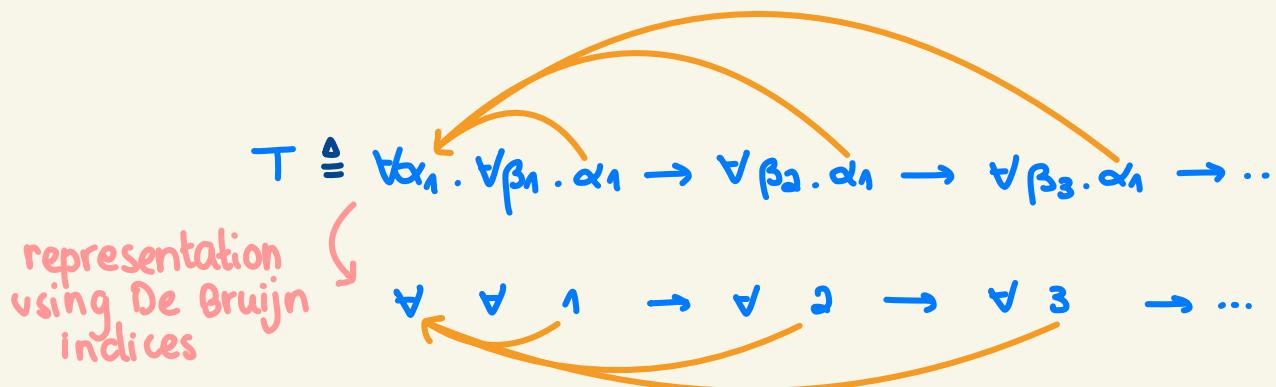
$$T \triangleq \forall \alpha_1. \forall \beta_1. \alpha_1 \rightarrow \forall \beta_2. \alpha_1 \rightarrow \forall \beta_3. \alpha_1 \rightarrow \dots$$

$$\hookrightarrow X \triangleq \forall \alpha. Y$$

$$Y \triangleq \forall \beta \alpha \rightarrow Y$$

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A A 1 → A 2 → A 3 → ...  
↳ NO FINITE REPRESENTATION

## WHAT WE LEARNED

- Equations are more expressive than  $\mu$ -types

$$U \triangleq \forall \alpha_1. \forall \beta_1. \alpha_1 \rightarrow \forall \alpha_2. \beta_1 \rightarrow \forall \beta_2. \alpha_2 \rightarrow \forall \alpha_3. \beta_2 \rightarrow \dots$$

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The diagram illustrates a sequence of type annotations:  $\forall \alpha_1. \forall \beta_1. \alpha_1 \rightarrow \forall \alpha_2. \beta_1 \rightarrow \forall \beta_2. \alpha_2 \rightarrow \forall \alpha_3. \beta_2 \rightarrow \dots$ . Orange curved arrows connect the  $\beta$  variables ( $\beta_1, \beta_2, \dots$ ) to their preceding  $\alpha$  variables ( $\alpha_1, \alpha_2, \dots$ ), indicating a dependency or derivation relationship between them.

## WHAT WE LEARNED

- Equations are more expressive than  $\mu$ -types

$$U \stackrel{\Delta}{=} \forall \alpha_1. \forall \beta_1. \alpha_1 \rightarrow \forall \alpha_2. \beta_1 \rightarrow \forall \beta_2. \alpha_2 \rightarrow \forall \alpha_3. \beta_2 \rightarrow \dots$$

$$\hookrightarrow \forall \alpha. X$$

$$X \doteq \forall \beta. \alpha \rightarrow \forall \alpha. \beta \rightarrow X \quad (*)$$

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↳  $\forall \alpha. X$

$$X \doteq \forall \beta. \alpha \rightarrow \forall \alpha. \beta \rightarrow X \quad (*)$$

↳ CANNOT BE WRITTEN USING CONVENTIONAL  $\mu$ -TYPES

WHAT WE  
DID WITH  
WHAT WE  
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- We kept using equations

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$$U \stackrel{\Delta}{=} \forall \alpha_1. \forall \beta_1. \alpha_1 \rightarrow \forall \alpha_2. \beta_1 \rightarrow \forall \beta_2. \alpha_2 \rightarrow \forall \alpha_3. \beta_2 \rightarrow \dots$$

↳  $\forall \alpha. X$

↙ \* after expansion

$$X \doteq \forall \beta. \alpha \rightarrow \forall \alpha. \beta \rightarrow X \xrightarrow{\text{renaming}} X \doteq \forall \beta. \alpha \rightarrow \forall \alpha'. \beta \rightarrow X$$

BACK TO  
OUR GOAL

Full polymorphism

HeteroTree  $\doteq \& \{ \text{Leaf} . \text{skip},$

Node : HeteroTree ;  $(\forall \alpha : T. ?\alpha)$  ;  
Hetero Tree  
}

 Kind s

 Kinds

TYPE EQUIVALENCE  
WITH DE BRUIJN  
INDICES

$$\frac{T \cong U}{\forall^k.T \cong \forall^k.U}$$

TYPE EQUIVALENCE  
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$$\frac{T \simeq U}{\forall^k.T \simeq \forall^k.U}$$

TYPE EQUIVALENCE  
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$$\frac{T \simeq U[\alpha/\beta]}{\forall \alpha : K.T \simeq \forall \beta : K.U}$$

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but  
we do not  
want to make  
these substitutions  
on the fly

CANONICAL  
RENAMING

$$N(\alpha) = \alpha$$

## CANONICAL RENAMING

$$N(\alpha) = \alpha$$

$$N(\forall \alpha : K. T) = \forall \alpha_i : K. N(T[\alpha_i / \alpha])$$

fresh ↴  
variable

$\alpha_i$  = first variable not free in  $\forall \alpha : K. T$

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e.g.,

$$N(\forall \alpha : T. \forall \beta . s . \beta)$$

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$$N(\forall \alpha : T. \forall \beta : s. \beta) = \forall \gamma : T. N(\forall \beta : s. \beta [\gamma / \alpha]) , \text{ where } \gamma < \dots$$

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$$\begin{aligned} N(\forall \alpha : T. \forall \beta : s. \beta) &= \forall \gamma : T. N(\forall \beta : s. \beta [\gamma / \alpha]) \\ &= \forall \gamma : T. N(\forall \beta : s. \beta) \end{aligned}$$

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$$= \forall \gamma : T. N(\forall \alpha : s. \beta)$$

$$= \forall \gamma : T. \forall \alpha : s. \beta$$

bound here

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bound here

The canonical renaming uses the least amount of variable names possible.

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$$N(\forall \alpha : K. T ; U) = \forall \alpha_i : K. (N(T[\alpha_i / \alpha]) ; N(U))$$

where  $\alpha_i = \text{first}(\forall \alpha : K. T ; U)$

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→ no need to introduce other rules

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MERCI !